

CONTRACTIONS AND EXPANSION

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ABSTRACT. Let $A \subseteq \mathbb{R}$ be a finite set and let $K \geq 1$ be a real number. Suppose that for each $a \in A$ we are given an injective map $\phi_a : A \rightarrow \mathbb{R}$ which fixes a and contracts other points towards it in the sense that $|a - \phi_a(x)| \leq \frac{1}{K}|a - x|$ for all $x \in A$, and such that $\phi_a(x)$ always lies between a and x . Then

$$|\bigcup_{a \in A} \phi_a(A)| \geq \frac{K}{10}|A| - O_K(1).$$

An immediate consequence of this is the estimate $|A + K \cdot A| \geq \frac{K}{10}|A| - O_K(1)$, which is a slightly weakened version of a result of Bukh.

To the memory of Yayha Hamidoune

1. INTRODUCTION

In this short note we consider the behaviour of a set $A \subseteq \mathbb{R}$ under a collection of maps $\phi_a : A \rightarrow \mathbb{R}$. Let $K \geq 1$ be a parameter. We will assume that these maps have the following properties:

- (i) ϕ_a is injective;
- (ii) $\phi_a(a) = a$;
- (iii) ϕ_a is a K -contraction in the sense that $|a - \phi_a(x)| \leq \frac{1}{K}|a - x|$ for all $x \in A$;
- (iv) $\phi_a(x)$ lies between a and x .

Theorem 1. *Suppose that $A \subseteq \mathbb{R}$ is a finite set of size n and that we have maps ϕ_a as above. Then*

$$|\bigcup_{a \in A} \phi_a(A)| \geq \frac{1}{8}Kn(1 - n^{-1/K^2}).$$

Remark. The bound we have given here looks a little odd, but it is convenient for our proof. Note that it is at least $\frac{1}{10}Kn - O(e^{CK^2})$, a slightly more precise version of the bound stated in the abstract.

An immediate corollary of this theorem is the following. Here, $A + K \cdot A := \{a + Ka' : a, a' \in A\}$.

Corollary 1. *Suppose that $A \subseteq \mathbb{R}$ is a finite set and that $K \geq 1$ is a real number. Then $|A + K \cdot A| \geq \frac{1}{10}K|A| - O(e^{CK^2})$.*

Proof. Simply apply the theorem with $\phi_a(x) := (x + Ka)/(K + 1)$. These maps obviously verify (i) – (iv) above. \square

We note that Bukh [1] established a much more precise result when $K \in \mathbb{Z}$, namely that $|A + K \cdot A| \geq (K + 1)|A| - o(|A|)$. Assuming that K is an integer should not make things any easier, and furthermore our approach would appear not to generalise to the more general sums of dilates $\lambda_1 \cdot A + \dots + \lambda_t \cdot A$ considered by Bukh. Let us also note that Cilleruelo, Hamidoune and Serra [3] obtained an extremely precise result when K is prime, establishing that $|A + K \cdot A| \geq (K + 1)|A| - \lceil K(K + 2)/4 \rceil$ for $|A| \geq 3(K - 1)^2(K - 1)!$.

2. PROOF OF THE MAIN THEOREM

In this section we prove Theorem 1. Let $F(n)$ be the minimum size of $\bigcup_{a \in A} \phi_a(A)$ over all sets A of size n . We will obtain a lower bound for $F(n)$ in terms of the values of $F(n')$, $n' < n$; we may then proceed by induction.

We will use the (obvious) *convexity* property of maps ϕ_a satisfying (i) – (iv) above, namely that $\phi_a(I) \subseteq I$ for any interval I containing a .

We clearly have $F(1) = 1$, so suppose that $A \subseteq \mathbb{R}$ is a set of size $n \geq 2$. We may rescale so that the extreme points of A are 0 and 1. Suppose that there is some $a_* \in A$ such that $|A \cap [a_* - 1/K, a_* + 1/K]| \leq 6n/K$. Write $A_1 := A \cap [0, a_* - 1/K)$ and $A_2 := A \cap (a_* + 1/K, 1]$, and set $n_1 := |A_1|$, $n_2 := |A_2|$. Then ϕ_{a_*} contracts all of A into the interval

$[a_* - 1/K, a_* + 1/K]$. By induction and the convexity property we have

$$\begin{aligned} F(n) &\geq \left| \bigcup_{a \in A_1} \phi_a(A_1) \right| + |A| + \left| \bigcup_{a \in A_2} \phi_a(A_2) \right| \\ &\geq F(n_1) + n + F(n_2). \end{aligned} \quad (2.1)$$

Note that $n_1 + n_2 \geq (1 - 6/K)n$.

Alternatively, suppose there is no such a_* . Obviously $A = A \cap \bigcup I_a$, where $I_a = [a - 1/K, a + 1/K]$. We may pass to disjoint subcollections $\bigcup_{a \in S_1} I_a$ and $\bigcup_{a \in S_2} I_a$ whose union is $\bigcup_{a \in A} I_a$ (cf. [2]). By assumption, $|A \cap I_a| > 6n/K$, and therefore $|S_1|, |S_2| < K/6$. It follows that A is covered by $< K/3$ intervals of length $2/K$, and hence there is some $a^* \in A$, $a^* < 1$, such that A is disjoint from $(a^*, a^* + 1/K]$. Set $A_1 := A \cap [0, a^*]$ and $A_2 := (a^* + 1/K, 1]$, and set $n_1 := |A_1|$, $n_2 := |A_2|$; note that $n_1 + n_2 = n$. Note also that $\phi_{a^*}(A_2) \subseteq (a^*, a^* + 1/K]$; here, we make crucial use of property (iv), which asserts that $\phi_{a_*}(x)$ lies between a_* and x .

By the convexity property and the above observations,

$$F(n) \geq \left| \bigcup_{a \in A_1} \phi_a(A_1) \right| + |A_2| + \left| \bigcup_{a \in A_2} \phi_a(A_2) \right| \geq |A_2| + F(n_1) + F(n_2).$$

Note, however, that A_2 contains 1 and hence $A \cap I_1$, a set of size $> 6n/K$. Therefore

$$F(n) \geq \frac{6n}{K} + F(n_1) + F(n_2) \quad (2.2)$$

in this case.

We have established that for each n there are $n_1, n_2 < n$ such that either (2.1) or (2.2) holds. That is,

$$\begin{aligned} F(n) &\geq \min \left(\min_{\substack{n_1, n_2 < n \\ n_1 + n_2 = n}} (F(n_1) + F(n_2) + \frac{6n}{K}), \right. \\ &\quad \left. \min_{\substack{n_1, n_2 < n \\ n_1 + n_2 \geq n(1 - 6/K)}} (F(n_1) + F(n_2) + n) \right). \end{aligned} \quad (2.3)$$

It remains to verify, by induction on n , that $F(n) \geq \frac{1}{8}Kn(1 - n^{-1/K^2})$.

Dealing with the first inequality immediately reduces to showing that

$$\frac{48n}{K^2} \geq n_1^{1-1/K^2} + n_2^{1-1/K^2} - n^{1-1/K^2}$$

whenever $n_1 + n_2 = n$. But the largest value of the right-hand side is never more than when $n_1 = n_2 = n/2$, and it is then enough to note that $48/K^2 \geq 2^{1/K^2} - 1$.

Checking the second inequality is even easier and amounts, using the inequality $n_1 + n_2 \geq (1 - 6/K)n$, to establishing that

$$\frac{2n}{K} \geq n_1^{1-1/K^2} + n_2^{1-1/K^2} - n^{1-1/K^2}$$

under the assumption that $n_1 + n_2 \leq n$. This completes the proof of Theorem 1. \square

3. SOME FURTHER REMARKS.

Condition (iv) above, that $\phi_a(x)$ always lies between a and x , may seem a little unnatural. In this section we note that Theorem 1 fails without this assumption.

Proposition 1. *Let $K \geq 1$ be arbitrary. Then there are arbitrarily large finite sets A together with collections of maps $\phi_a : A \rightarrow \mathbb{R}$ satisfying (i), (ii) and (iii) above but such that*

$$|\bigcup_{a \in A} \phi_a(A)| = 2|A|.$$

Proof. Assume that $K > 10^4$ (say). Consider the set

$$A := \{\varepsilon_n(4K)^{-n} + \cdots + \varepsilon_1(4K)^{-1} : \varepsilon_1, \dots, \varepsilon_n \in \{0, 1\}\},$$

and define

$$\tilde{A} := \{\varepsilon_{n+1}(4K)^{-n-1} + \cdots + \varepsilon_1(4K)^{-1} : \varepsilon_1, \dots, \varepsilon_{n+1} \in \{0, 1\}\}.$$

Obviously $|\tilde{A}| = 2|A|$. There is an obvious map $\phi_0 : A \rightarrow \tilde{A}$ defined by $\phi_0(x) = x/4K$; this clearly satisfies $\phi_0(0) = 0$, and is a $1/4K$ -contraction. Now for each $x \in A$ there is a bijection $\psi_x : \tilde{A} \rightarrow \tilde{A}$ with

$\psi_x(A) = A$, $\psi_x(0) = x$, and which is 2-biLipschitz in the sense that

$$\frac{1}{2}|t - t'| \leq |\psi_x(t) - \psi_x(t')| \leq 2|t - t'|$$

for all $t, t' \in \tilde{A}$. Such a map may be constructed by viewing \tilde{A} as the set of vertices of a binary tree of depth $n + 1$ and then applying a suitable binary tree automorphism. The distance between two nodes is determined, up to a factor of at most 1.01, by the point in the tree at which they branch (or equivalently the smallest m for which $\varepsilon_m \neq \varepsilon'_m$), and this is preserved by any tree automorphism.

Now define $\phi_x : A \rightarrow \tilde{A}$ by $\phi_x := \psi_x \circ \phi_0 \circ \psi_x^{-1}$. This is well-defined because $\psi_x^{-1}(A) = A$, and ϕ_0 is defined on A . Obviously $\phi_x(x) = x$. Furthermore ϕ_x , being a composition of a $1/4K$ -contraction and two 2-biLipschitz maps, is a $1/K$ -contraction. Each ϕ_x is injective, and finally $|\bigcup_{a \in A} \phi_a(A)| = |\tilde{A}| = 2|A|$. This completes the proof. \square

It is clear that one cannot remove the factor of 2 in this proposition if $K \geq 2$. Indeed if a_1, a_2 are the extreme points of A then $\phi_{a_1}(A)$ and $\phi_{a_2}(A)$ are disjoint and both have cardinality $|A|$.

Finally let us note that one should not expect a version of Corollary 1 with polynomial dependencies throughout. Indeed the set

$$A := \{n_0 + n_1K + \cdots + n_{d-1}K^{d-1} : n_0, \dots, n_{d-1} \in [N]\}$$

has $|A + K \cdot A| \leq 2^d N^{d+1} = 2^d N |A|$. Taking $d \sim \frac{1}{2} \log_2 L$ and $N := L/2^d$ we may ensure that $|A + K \cdot A| \leq L|A|$ whilst $|A| = N^d \geq e^{c(\log L)^2}$. In particular, if $L \sim e^{(\log K)^{3/4}}$ then we have $|A + K \cdot A| \leq K^{o(1)}|A|$ whilst $|A| \geq K^{c(\log K)^{1/2}}$.

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